Reconstruction of photon statistics using low performance photon counters

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The output of a photodetector consists of a current pulse whose charge has the statistical distribution of the actual photon numbers convolved with a Bernoulli distribution. Photodetectors are characterized by a nonunit quantum efficiency, *i.e.* not all the photons lead to a charge, and by a finite resolution, *i.e.* a different number of detected photons leads to a discriminable values of the charge only up to a maximum value. We present a detailed comparison, based on Monte Carlo simulated experiments and real data, among the performances of detectors with different upper limits of counting capability. In our scheme the inversion of Bernoulli convolution is performed by maximum-likelihood methods assisted by measurements taken at different quantum efficiencies. We show that detectors that are only able to discriminate between zero, one and more than one detected photons are generally enough to provide a reliable reconstruction of the photon statistics for single-peaked distributions, while detectors with higher resolution limits do not lead to further improvements. In addition, we demonstrate that, for semiclassical states, even on/off detectors are enough to provide a good reconstruction. Finally, we show that a reliable reconstruction of multi-peaked distributions requires either higher quantum efficiency or better capability in discriminating high number of detected photons.

I. INTRODUCTION

Reconstruction of the photon statistics, ϱ_n, of optical states provides fundamental information on the nature of any optical field and finds relevant applications in foundations of quantum mechanics, quantum state engineering by postselection [1], quantum information [2], and quantum metrology. Indeed, detectors with the capability of counting photons [3, 4] are currently under investigation. Among these, photomultiplier tubes (PMT's) [5] and hybrid photodetectors [4, 6] are promising devices, though they have the drawback of a low quantum efficiency. On the other hand, solid state detectors with internal gain are still under development. Highly efficient thermal detectors have also been used as photon counters, though their operating conditions are still extreme to allow common use [7, 8]. The advent of quantum tomography provided an alternative method to measure photon number distributions [9]. However, tomography needs the implementation of homodyne detection, which involves challenging mode matching, especially in the case of pulsed optical fields.

In principle, in a photodetector each photon ionizes a single atom, and the resulting charge is amplified to produce a measurable pulse. In practice, however, available photodetectors are usually characterized by a quantum efficiency lower than unity, which means that only a fraction of the incoming photons leads to an electric pulse. If the resulting current is proportional to the incoming photon flux we have a linear photodetector. This is, for example, the case of the high-flux photodetectors used in homodyne detection. On the other hand photodetectors operating at very low intensities usually resort to avalanche process or very high amplification in order to transform a single ionization event into a recordable pulse. Due to the gain instability in the process of amplification it is generally difficult to discriminate the number of detected photons as far as this number becomes larger.

As a consequence the output of a photodetector consists of a current pulses whose charge statistics is a generalized Bernoulli convolution of the actual photon distribution with the possibility to perform a discrimination between the number of detected photons only up to a finite number. We define M as the maximum number m of detected photons that can be distinguished from $m-1$, M represent the counting capability of the detectors: $i.e.$ detectors that works in Geiger mode have $M=1$ because form the output is possible to discriminate from $m=1$, one detected photon, to $m=0$, dark, but is not possible to

discriminate the output for $m \geq 2$ from that for to $m = 1$. The inversion of such a convolution may be performed in several ways, though, in general it is possible only when the quantum efficiency is larger than $\eta = 0.5$ [10], and it is inherently inefficient, as it requires a large data sample to provide a reliable result. On the other hand, maximum-likelihood methods assisted by measurements taken at different quantum efficiencies have been proved to be both effective and statistically reliable [11, 12, 13, 14]. In particular, it is possible to obtain a method to reconstruct ϱ_n *without any a priori information* on the state of light under investigation. An alternative approach is based on the so called *photon chopping* [15], *i.e.* on a network of beam splitter followed by an array of on/off detectors. Photon chopping allows to send at most one photon on each detector, however, with the drawback of increasing the overall complexity of the detection scheme.

In this paper, we extend previous analysis on maximum-likelihood on/off-based schemes [12, 13], and present a detailed comparison among the performances of detectors characterized by realistic quantum efficiencies and different values of M. The comparison is made using real data and by an extensive set of Monte Carlo simulations, performed on different states of the radiation field including quantum and semiclassical states associated to single-peaked as well as multi-peaked photon distributions. The reconstruction is obtained by maximum-likelihood methods assisted by measurements taken at different quantum efficiencies. In particular, as the statistics of the detected photons is linearly dependent on the photon distribution (see below), the inversion can be well approximated by using the Expectation-Maximization algorithm (EM) which leads to an effective and reliable iterative solution.

Our results indicates that detectors with $M = 2$ discriminating between zero, one and more than one photon, are generally enough to provide a reliable reconstruction of the photon statistics for single-peaked distributions, while detectors with higher value of M do not lead to further improvements. On the other hand, multi-peaked distributions requires a higher quantum efficiency whereas for semiclassical states even on/off detectors are enough to provide a good reconstruction.

The paper is structured as follows. In the next Section we describe the statistics of the detected photons and illustrate the reconstruction algorithm. In Section III we report the results of a set of Monte Carlo simulated experiments performed on different kinds of signals, whereas in Section IV we report experimental results for coherent signals. Section V closes the paper with some concluding remarks.

II. STATISTICS OF DETECTED PHOTONS AND RECONSTRUCTION ALGORITHM

Using a photodetector with quantum efficiency η and $M = \infty$, that corresponding to an unlimited counting capability, the probability of obtaining m detected photons at the output is given by the convolution

$$p_\eta(m) = \sum_{n=m}^{\infty} A_\nu(m,n)\varrho_n \ , \tag{1}$$

where $\varrho_n = \langle n|\varrho|n\rangle$ is the actual photon statistics of the signal under investigation and

$$A_\eta(m,n) = \binom{n}{m} (1-\eta)^{n-m}\eta^m \ . \tag{2}$$

If M have a finite value only $M+1$ outcomes are possible, which occur with probabilities

$$q_\eta^m = p_\eta(m) \quad m = 0, ..., M-1 \tag{3}$$

$$q_\eta^M = \sum_{m=M}^{\infty} p_\eta(m) = 1 - \sum_{m=0}^{M-1} q_\eta^m \ . \tag{4}$$

Once the value of the quantum efficiency is known, Eqs. (3) and (4) provide M relations among the statistics of detected photons and the actual statistics of photons. At first sight the statistics of a detector with low

counting capability (M in the range $M = 1, ..., 6$) appears to provide quite a scarce piece of information about the state under investigation. However, if the statistics about $\{q_\eta^m\}$ is collected for a suitably large set of efficiency values, then the information is enough to reconstruct the whole photon distribution ϱ_n of the signal, upon a suitable truncation at N of the Hilbert space. We adopt the following strategy: by placing in front of the detector K filters with different transmissions, we may perform the detection with K different values η_ν, $\nu = 1...K$, ranging from zero to a maximum value $\eta_K = \eta_{\max}$ equal to the nominal quantum efficiency of the detector. By denoting the probability of having m detected photons in the experiment with quantum efficiency $\eta = \eta_\nu$ by $q_\nu^m \equiv q_{\eta_\nu}^m$ we can rewrite Eqs. (3) and (1) as

$$q_\nu^m = \sum_{n=m}^{\infty} A_\nu(m,n)\varrho_n \; , \tag{5}$$

where $\nu = 1, ..K$ and $m = 0, .., M-1$. Let us now suppose that the ϱ_n's are negligible for $n > N$ and that the η_ν's are known, then $\forall \eta_\nu$ Eq. (5) may be rewritten a finite sum over n from m to N. Overall, we obtain a finite linear system with $K \times M$ equations in the N unknowns $\{\varrho_n\}$. Unfortunately, the reconstruction of ϱ_n by matrix inversion cannot be used in practice since it would require an unreasonable number of experimental runs to assure the necessary precision [11, 12]. This problem can be circumvented by considering Eqs. (5) as a statistical model for the parameters $\{\varrho_n\}$ to be solved by maximum-likelihood (ML) estimation. The likelihood functional is given by the global probability of the sample *i.e*

$$\mathsf{L} = \prod_{\nu=1}^{K} \prod_{m=0}^{M-1} (q_\nu^m)^{n_{m\nu}} \quad . \tag{6}$$

The ML estimates of $\{\varrho_n\}$ are the values that maximizes L. In Eq. (6) $n_{m\nu}$ denotes the number of events "m detected photons" obtained with quantum efficiency η_ν. The maximization of L under the conditions $\varrho_n \geq 0$, $\sum_n \varrho_n = 1$, can be well approximated by using the expectation-maximization (EM) algorithm [16, 17, 18], which leads to the iterative solution

$$\begin{aligned} \varrho_n^{(i+1)} =& \varrho_n^{(i)} \left(\sum_{\nu=1}^{K} \sum_{m=0}^{M} A_\nu(m,n) \right)^{-1} \\ & \times \sum_{\nu=1}^{K} \sum_{m=0}^{M} A_\nu(m,n) \, \frac{f_\nu^m}{q_\nu^m[\{\varrho_n^{(i)}\}]} \; . \end{aligned} \tag{7}$$

In Eq. (7) $\varrho_n^{(i)}$ denotes the n-th element of reconstructed statistics at the i-th step, $q_\nu^m[\{\varrho_n^{(i)}\}]$ the theoretical probabilities as calculated from Eq. (5) at the i-th step, whereas $f_\nu^m = n_{m\nu}/n_\nu$ represents the frequency of the event "m detected photons" with quantum efficiency η_ν, being n_ν the total number of runs performed with $\eta = \eta_\nu$. In the following we assume equal n_ν, $\forall \nu$.

Eq. (7) provides a solution once an initial distribution is chosen. In the following we will always consider an initial uniform distribution $\varrho_n^{(0)} = (1+N)^{-1} \; \forall n$ in the truncated Fock space $n = 0, ..., N$. Results from Monte Carlo simulated experiments shows that any distribution with $\varrho_n^{(0)} \neq 0 \; \forall n$ performs equally well, *i.e.* the choice of the initial distribution does not alter the quality of reconstruction, though it may slightly affect the convergence properties of the algorithm.

III. MONTE CARLO SIMULATED EXPERIMENTS

In this Section we report the photon statistics for different states of a single-mode radiation field as obtained by ML reconstruction from Monte Carlo simulated photodetection performed with the same low quantum efficiency and different resolution threshold. We consider semiclassical states as well as highly nonclassical states and show results for different values of the maximal quantum efficiency $\eta_{\max}$.

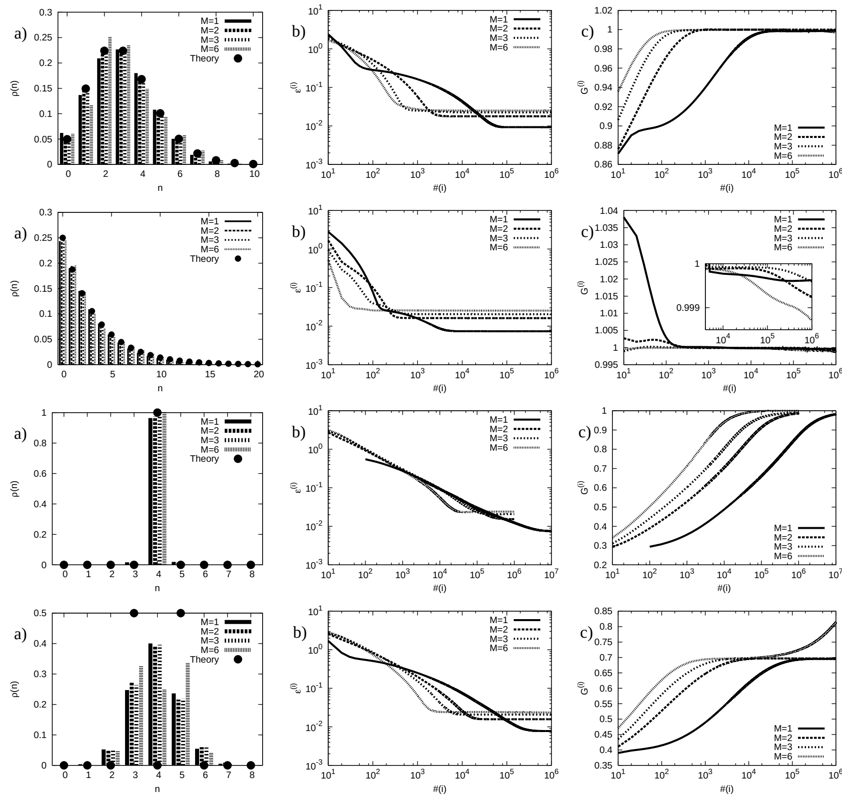

FIG. 1: Maximum-likelihood reconstruction of photon statistics using photodetectors with different counting capability (different values of M) and maximum quantum efficiency $\eta_{max} = 0.2$. First line: photon distribution of a coherent state with $\langle a^\dagger a\rangle = 3$; panel a) the reconstructed ρ_n; b): total absolute error $\varepsilon^{(i)}$; according to Eq. (8) c): fidelity $G^{(i)}$ according to Eq. (9). Results are reported for $M = 1$ (on/off detector), $M = 2$, $M = 3$ and $M = 6$. We use $K = 30$ different quantum efficiencies $\eta = \eta_\nu$, uniformly distributed in $[\eta_{max}/K, \eta_{max}]$. The Hilbert space have been truncated at $N = 30$ and $n_\nu = 10^6$ runs have been performed for each value η_ν. The last iteration is $\#(i_L) = n_\nu$. Other lines: same as first line for thermal state with $\langle a^\dagger a\rangle = 3$, Fock state with $n = 4$; superposition of 2 Fock states $\frac{1}{\sqrt{2}}(|3\rangle + |5\rangle)$. The inset of panel c) in the third line shows the last iterations $\#(i)$ in an expanded scale.

In order to assess our results we introduce two figures of merit, to measure the reliability and the accuracy of the method respectively. Since the solution of the ML estimation is obtained iteratively, the most important aspect to keep under control is the convergence of the algorithm. A suitable parameter to evaluate the degree of convergence at the i-th iteration is the total absolute error

$$\varepsilon^{(i)} = \sum_{\nu=1}^{K} \sum_{m=0}^{M} \left| f_\nu^m - q_\nu^m[\{\varrho_n^{(i)}\}] \right| . \tag{8}$$

Indeed, the total error measures the distance of the probabilities $q_\nu^m[\{\varrho_n^{(i)}\}]$, as calculated at the i-th iteration, from the actual experimental frequencies and thus, besides convergence, it quantifies how the estimated distribution reproduces the experimental data. The total distance is a decreasing function of the number of iterations. Its stationary value is proportional to the accuracy of the experimental frequencies $\{f_\nu^m\}$. For finite data sample this value is of order $1/\sqrt{n_\nu}$ for each value of η_ν. As a measure of accuracy at the i-th step we adopt the so-called fidelity between probability distributions

$$G^{(i)} = \sum_{n=0}^{N-1} \sqrt{\varrho_n^{(i)} \varrho_n} . \tag{9}$$

In Ref. [13] it was shown that using on/off detection (*i.e.* $M = 1$) a reliable reconstruction scheme may be obtained. In this paper, our aim is to check whether a higher value of M leads to some advantages, either in terms of accuracy or convergence.

Fig. 1 summarizes results for different states and values of M, assuming $\eta_{max} = 0.2$ as the maximum value of the quantum efficiency. In order to compare the performances achievable by different M one has first to chose a criterion to stop the iterative algorithm. A natural criterion would be that of stopping the iteration when the total absolute error $\varepsilon^{(i)}$ converges, *i.e* when the rate of its variation falls below a threshold and becomes negligible. This is also motivated by the behavior of $\varepsilon^{(i)}$ versus the number of iterations (see central column in Fig. 1): a rapid fall followed by a plateau. However, the most convenient value for the threshold unavoidably depends on the shape of the unknown distribution $\varrho_{(n)}$. Therefore, in the absence of any *a priori* information, this simple criterion should be supplemented by some additional recipes. We found numerically that a suitable criterion, valid for any value of M and, in the average, for any class of states, is the following: if the total absolute error appears to converge, then stop the algorithm at a number of iterations $\#(i) = n_\nu$ equal to the number of measurements taken at each value of the quantum efficiency, otherwise stop the algorithm as soon as you see convergence. In cases when $\varepsilon^{(i)}$ converges for a number of iteration $\#(i) < n_\nu$ it is not convenient to stop the algorithm, since it may further increase the quality of the reconstruction, see for example $G^{(i)}$ in the fourth line of Fig 1. On other hands, if $\#(i)$ goes far beyond the condition of convergence of $\varepsilon^{(i)}$, then the algorithm may lose precision due to the noise of the data f_ν^m (see inset in the second line of Fig. 1). The choice of the threshold at $\#(i) = n_\nu$ fits with other two independent facts. On one hand, we performed an extensive set of simulations and no considerable reduction of $G^{(i)}$ was observed before the convergence condition $\#(i) = n_\nu$. On the other hand, we applied the reconstruction algorithm to the *exact* on/off frequencies (not to simulated data) and no reduction of $G^{(i)}$ was observed for increasing $\#(i)$, thus confirming the data-noise origin of precision loss.

Let us now illustrate our results about the accuracy of the reconstructions. As mentioned above we performed simulated experiments, on semiclassical states like coherent and thermal ones as well as highly nonclassical states such Fock states, by using three values for the maximum quantum efficiency $\eta_{\max} = 0.2, 0.5, 0.8$. Simulations have been performed by using $n_\nu = 10^4$ data at each value of the quantum efficiency. This value of n_ν is relatively small and well within the realm of quantum optical experiment. In a real experiment performed under the same conditions, n_ν of this order would allow on-line reconstruction of the photon statistics. Simulations performed using different values of the parameters lead to similar conclusion.

In Fig. 2 we show the fidelity of reconstruction $G^{(i)}$ at the last iterations, $G^{(i_L)}$, for different input signals and different resolution thresholds as a function of the average number of photons of the signal. As it is apparent from the plots, for low quantum efficiency and semiclassical states on/off detectors ($M = 1$) are enough to achieve a good reconstruction. On the other hand, the accuracy for nonclassical states largely improves for $M \geq 2$. For higher values of the quantum efficiency (the middle and the right plots) on/off detectors becomes sufficient for a good reconstruction also for nonclassical states having single-peaked distributions. Notice that a higher resolution, however with a low value of η_{max}, does not guarantee a good reconstruction. Also notice that the results reported in Fig. 2 has been obtained by stop the algorithm according to the prescriptions mentioned above. We therefore do not expect that the fidelity is optimal for *any* state. Indeed the fidelity $G^{(i_L)}$ in panel C5) and C8) of Fig. 2 slightly decreases for high number of photons, though remaining close to unit value. The decreases in panels T2), T5) and T8) of Fig. 2 for greater mean values is due to the choice of a small dimension of the truncated Hilbert space: by increasing N the trend disappear. The low values of $G^{(i_L)}$ for the superposition states are due to the noise in f_ν^m, also for the high quantum efficiencies. Indeed, by increasing n_ν the reconstructions greatly improves; besides, for high quantum efficiency the detectors perform good reconstructions also for $M = 0$, see the inset of panel S2) in Fig. 2. The error bars in Fig. 2 are standard deviations of $G^{(i_L)}$ as calculated from 40 different Monte Carlo runs. They may appear large, but this is due to the high value of $G^{(i_L)}$, which in turn determines the

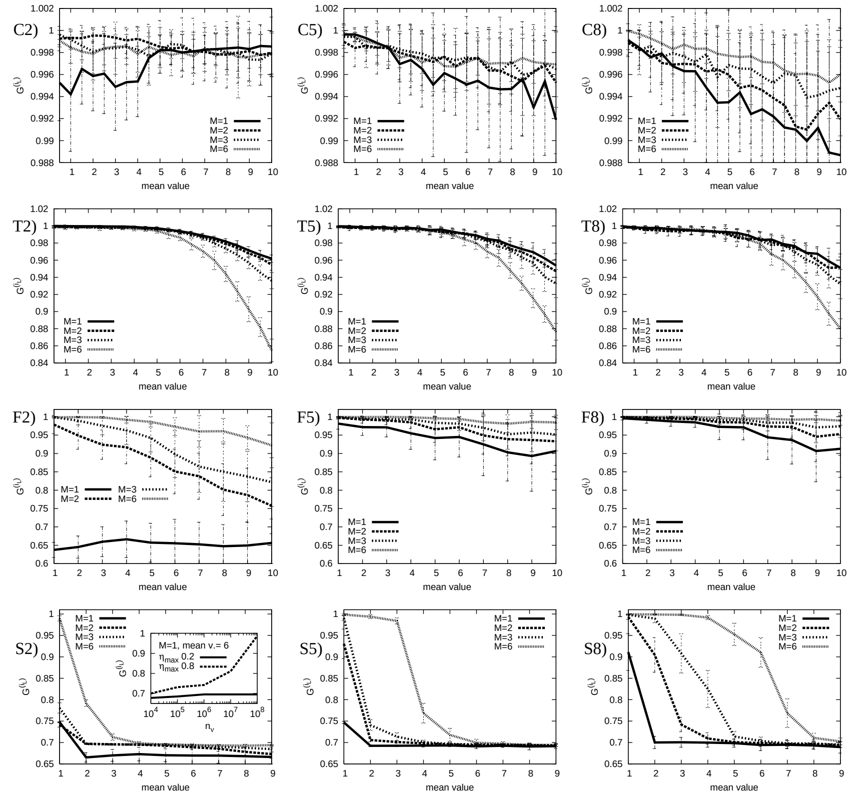

FIG. 2: Fidelity parameter at the last iteration, $G^{(i_L)}$, as a function of $\langle a^\dagger a\rangle$ for 4 different statistics of $\{\varrho_n\}$: coherent, panels labeled C); thermal, label T); Fock state, label F); superposition of 2 Fock states, $\frac{1}{\sqrt{2}}(|\langle a^\dagger a\rangle - 1\rangle + |\langle a^\dagger a\rangle + 1\rangle)$, label S). The reconstructions have been performed using $\eta_{max} = 0.2, 0.5$ and 0.8, panels labeled with 2), 5) and 8) respectively; $n_\nu = 10^4$ number of runs at each for every η_ν. According to the rule given in the text, the last iteration performed is $\#(i_L) = n_\nu$ except for coherent state with $M = 1$ and $\langle a^\dagger a\rangle = 1...4$ for which $\#(i_L) = 10^5$, and for the superpositions state. Inset of S2) show $G^{(i_L)}$ for $\frac{1}{\sqrt{2}}(|5\rangle + |7\rangle)$, M=0, quantum efficiency 0.2 and 0.8 in function of n_ν. Every point of the curves is the mean of 40 simulations, the error bars represent the standard deviations. The other parameters are the same as in fig.1.

scale of the plot. Look at panels S)' for comparison.

IV. EXPERIMENTAL DATA

In order to confirm the Monte Carlo results for single-peaked distributions we have performed the reconstruction of the photon statistics of a coherent signal obtained from a Nd:YLF laser. The experimental data have been recorded with a hybrid photo detector, Hamamatsu H8236-40, placed on the second harmonics (523.5 nm) of a cw mode-locked Nd:YLF laser regeneratively amplified at a repetition rate of 5 kHz (High Q Laser).

The frequencies f_ν^m, until $m = 3$, have been extrapolated from the response of the detector with the following procedure. At first a Gaussian best-fit of the response peaks at 0,..,4 photoelectrons has been performed [5], then the the frequencies has been obtained by choosing three thresholds, whose optimal choice turns out to be the mean point between photoelectron peaks[19]. After the frequencies f_ν^m have been obtained we used the algorithm of the previous Section. The results of the reconstruction are shown in Fig. 3, panel a). The reconstructed distribution at the last iteration $\varrho_n^{(i_L)}$ have been then compared with a Poissonian best-fit. As it is apparent from Fig. 3, the agreement between $\varrho_n^{(i_L)}$ and the best-fit is very good.

In addition, we notice that the shape of $G^{(i)}$ and $\varepsilon^{(i)}$ are very similar to those coming from simulation and that the $\varrho_n^{(i)}$ obtained by the experimental data with M=1 is close to the one obtained with M=2 and M=3, as predicted by simulations.

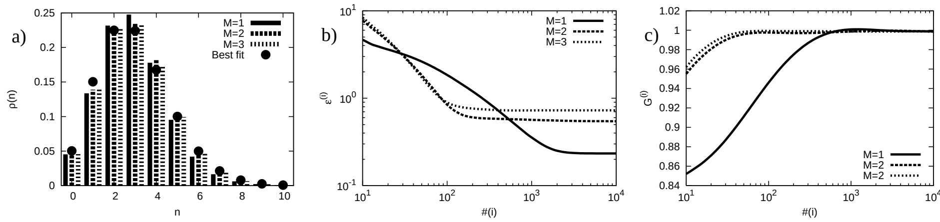

FIG. 3: Experimental reconstruction of the photon number distribution of a coherent state. a): reconstructed ρ_n and coherent best fit, $\langle a^\dagger a \rangle = 2.98$; b): total absolute error $\varepsilon^{(i)}$; c): fidelity $G^{(i)}$ calculated *a posteriori*. The reconstruction is performed with a hybrid photo detector operated by taking $M = 1$ (on/off detector), $M = 2$ and $M = 3$; $K = 100$ different quantum efficiencies $\eta = \eta_\nu$ distributed in $[0, \eta_{max}]$, $\eta_{max} = 0.4$; the Hilbert space is truncated at $N = 30$; $n_\nu = 10^4$ number of runs have been performed for each η; The algorithm is stopped at iteration $\#(i_L) = n_\nu$.

V. CONCLUSION

We have compared the reconstruction of the photon statistics as obtained applying ML algorithm to data coming from detectors with different counting capability and different quantum efficiencies. We found that using ML methods, detectors with high quantum efficiency does not need to have high counting capability, since on/off detection assisted by ML methods already provides good state reconstructions. On the other hand, a small quantum efficiency makes the counting capability a crucial parameter. Overall, our results indicate that development of future photodetectors may be focused on increasing the quantum efficiency rather than the counting capability.

VI. ACKNOWLEDGMENTS

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